Random Matrix Theory for Machine Learning

Part 3: Analysis of numerical algorithms

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https://random-matrix-learning.github.io
In this section:

- $A$ is typically a rectangular matrix with more rows than columns
- $W$ is a symmetric (square) matrix
- Often $W \propto A^T A$
Motivation: Average-case versus worst-case in high dimensions
In some very specific cases, the high-dimensionality of a given problem provides it with enough degrees of freedom to “conspire against” a given algorithm.
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For example, consider solving a $n \times n$ linear system $Wx = b$ using the conjugate gradient (CG) algorithm where

$$W = \begin{bmatrix}
    r & \sqrt{r} & \sqrt{r} & \sqrt{r} & \cdots & \sqrt{r} \\
    \sqrt{r} & 1 + r & \sqrt{r} & \sqrt{r} & \cdots & \sqrt{r} \\
    \sqrt{r} & \sqrt{r} & 1 + r & \sqrt{r} & \cdots & \sqrt{r} \\
    \sqrt{r} & \sqrt{r} & \sqrt{r} & 1 + r & \cdots & \sqrt{r} \\
    \sqrt{r} & \sqrt{r} & \sqrt{r} & \sqrt{r} & \cdots & 1 \\
    \sqrt{r} & \sqrt{r} & \sqrt{r} & \sqrt{r} & \sqrt{r} & \sqrt{r}
\end{bmatrix}, \quad b = \begin{bmatrix}
    1 \\
    0 \\
    \vdots \\
    \vdots \\
    0 \\
    0
\end{bmatrix}, \quad 0 < r < 1.
The CG algorithm is iterative and produces approximations $x_k$ that satisfy:

$$
x_k = \arg \min_{y \in \mathbb{R}^n} \left\{ (x - y)^T W (x - y)^T : y \in \text{span}\{b, Wb, \ldots, W^{k-1}b\} \right\}.
$$
The CG algorithm is iterative and produces approximations $\mathbf{x}_r$ that satisfy:

$$x_r = \arg \min_{y \in \mathbb{R}^n} \left\{ (x - y)^T W (x - y)^T : y \in \text{span}\{b, Wb, \ldots, W^{k-1}b\} \right\}.$$

It can be shown that for the above choice of $W, b$, and $1 \leq k < n$

$$\|b - Wx_r\|^2 = \left( \frac{1}{r} \right)^k \quad \text{but} \quad \|b - Wx_n\| = 0.$$
Issues with worst-case bounds in high dimensions

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The residuals (or norms of the gradient) appear to diverge exponentially before the iteration finally converges!

And as $n$ increases, this becomes worse. And a worst-case bound needs to account for this pathological example.
Introducing distributions

Instead, we may want to choose $W$ and $b$ to be random and consider

$$\mathbb{E}\|b - Wx_k\|^2.$$

If one chooses $W$ to be distributed according to the Wishart distribution, as $n \to \infty$,

$$\mathbb{E}\|b - Wx_k\|^2 = r^k + o(1), \quad r = r^{-1} = \lim_{n \to \infty} \frac{n}{d}.$$
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But a valid important open problem is: To model optimization in a ML context, what distribution is relevant for $W$?

This is an open problem. See Liao and Mahoney [2021] for work in this direction.
Main RMT tool: Matrix moments
Recall Cauchy’s integral formula: If $f$ is analytic in a sufficiently large region and $C$ is smooth, simple, closed curve then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} \, dz'.$$
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\]

Suppose the eigenvalues of an \( n \times n \) matrix \( W \) are enclosed by \( C \), then

\[
f(W) := UF(\Lambda)U^{-1} = \frac{1}{2\pi i} \int_C f(z) (zI_n - W)^{-1} \, dz.
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In particular,

\[
W^k = \frac{1}{2\pi i} \oint_C z^k(zI_n - W)^{-1} \, dz.
\]
Two consequences

$$\frac{1}{n} \text{tr} W^k = \frac{1}{2\pi i n} \int_C z^k \text{tr} (zI_n - W)^{-1} \, dz$$
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$$\frac{1}{n} \text{tr} W^k = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\mathcal{C}} z^k (z - \lambda_j)^{-1} \, dz$$
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\frac{1}{n} \text{tr} \ W^k = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{c} z^k (z - \lambda_j)^{-1} \, dz = \frac{1}{2\pi i} \int_{c} z^k \left\{ m_{\text{ESD}}(z) \right\} \, dz
\]

Stieltjes transform of \( n^{-1} \sum_j \delta \lambda_j \)
Two consequences

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Stieltjes transform of \( \frac{1}{n} \sum_j \delta \lambda_j \)

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Stieltjes transform of \( \sum_j w_j \delta \lambda_j \)

\[ \sum_j w_j \delta \lambda_j, \quad w_j = (v_j^T u)^2 \]
A main RMT takeaway:

Matrix moments $\Leftrightarrow$ Classical moments of ESD $\Leftrightarrow$ Contour integrals of Stieltjes transform
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\[ \frac{1}{n} \text{tr} \, W^k \approx \int_{\mathbb{R}} x^k \mu_{\text{ESD}}(dx) \approx \frac{1}{2\pi i} \int_{C} z^k m_{\text{ESD}}(z) \, dz \]
Analysis of matrix moments

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If $\mu_{\text{ESD}} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j(W)}$ then the first $\approx$ becomes $\approx$. 
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\[ \frac{1}{n} \text{tr} W^k \approx \int_{\mathbb{R}} x^k \mu_{\text{ESD}}(dx) = \frac{1}{2\pi i} \int_{\mathbb{C}} z^k m_{\text{ESD}}(z) \, dz \]

If \( \mu_{\text{ESD}} \) is the limiting ESD then the second \( \approx \) becomes \( = \).
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If $\mu_{\text{ESD}}$ is the limiting ESD then errors are typically on the order of $1/n$. 
A main RMT takeaway:

\[ u^T W^k u \approx \int_{\mathbb{R}} x^k \mu_{\text{ESD}}(dx) \approx \frac{1}{2\pi i} \int_{C} z^k m_{\text{ESD}}(z) \, dz \]
A main RMT takeaway:

\[ u^\top W^k u = \int_{\mathbb{R}} x^k \mu_{\text{ESD}}(dx) \approx \frac{1}{2\pi i} \int_{\mathcal{C}} z^k m_{\text{ESD}}(z) \, dz \]

If \( \mu_{\text{ESD}} = \sum_{j=1}^{n} w_j \delta_{\lambda_j}(w) \), \( w_j = (v_j^\top u)^2 \) for eigenvectors \( v_j \) then the first \( \approx \) becomes \( = \).
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\[ u^T W^k u \approx \int_{\mathbb{R}} x^k \mu_{\text{ESD}}(dx) = \frac{1}{2\pi i} \int_{\mathcal{C}} z^k m_{\text{ESD}}(z) \, dz \]

If \( \mu_{\text{ESD}} \) is the limiting ESD then errors are typically on the order of \( 1/\sqrt{n} \).
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$$u^T W^k u \approx \int_{\mathbb{R}} x^k \mu_{\text{ESD}}(dx) \approx \frac{1}{2\pi i} \int_{\mathbb{C}} z^k m_{\text{ESD}}(z) \, dz$$

If a statistic, (which might be the error encountered in, or the runtime of, an algorithm) depends strongly on these generalized moments, it may be analyzable directly using RMT.
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\[ u^T W^k u = \int_{\mathbb{R}} x^k \mu_{\text{ESD}}(dx) \approx \frac{1}{2\pi i} \int_{\mathcal{C}} z^k m_{\text{ESD}}(z) \, dz \quad \mu_{\text{ESD}} = \sum_{j=1}^d w_j \delta_{\lambda_j}(w) \]

**Theorem (Knowles and Yin [2017])**

For a large class of sample covariance matrices \( W \) there exists a deterministic measure \( \mu_{\text{SCM}} \) with Stieltjes transform \( m_{\text{SCM}} \) such that

\[
\Pr \left( \left| a^T (W - zI_n)^{-1} b - (a^T b) m_{\text{SCM}}(z) \right| \geq \|a\| \|b\| t \right) = O(n^{-D})
\]

for any \( D > 0 \), uniformly in a large subset of the complex plane.
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\[ \Pr \left( |m_{\text{ESD}}(z) - m_{\text{SCM}}(z)| \geq t \right) = O(n^{-D}) \]

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Resolvent estimates lead to moment estimates

\[ u^T W^k u \approx \frac{1}{2\pi i} \int_C z^k m_{ESD}(z) \, dz \approx \frac{1}{2\pi i} \int_C z^k m_{SCM}(z) \, dz = \int_{\mathbb{R}} x^k \mu_{SCM}(dx) \]
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\[ u^T P(W) u \approx \int_{\mathbb{R}} P(x) \mu_{SCM}(dx) \]
Algorithm halting times (runtimes)
Statistics of algorithm runtimes

Our abstract setup to analyze algorithms is as follows. Suppose first that there is a intrinsic notion of dimension $n$. 
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- Let $\mathcal{E}$ represent a distribution from which problems $P_n$ are drawn (e.g., a random matrix and vector for a linear system).
- The halting time is then defined as

$$T_{\mathcal{A}}(P_n, \varepsilon) = \min\{k : E_k(P_n; \mathcal{A}) < \varepsilon\}.$$
Probably the most famous, and maybe the most influential, instance of the probabilistic analysis of an algorithm, was the analysis of the simplex algorithm developed by Dantzig [1951] for linear programming (see also Dantzig [1990]).
Some history

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For many years after its inception the simplex method had no provable complexity guarantees. Indeed, with a fixed pivot rule, there typically exists a problem on which the simplex method takes an exponentially large number of steps.
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Despite the existence of other algorithms for linear programming with provable polynomial runtime guarantees, the simplex method persisted as the most widely used algorithm.
Some history

Borgwardt [1987] and, independently, Smale [1983] proved that under certain probabilistic assumptions and under certain pivot rules, the expected runtime of the simplex algorithm is polynomial:

$$\mathbb{E} T_{\text{Simplex}}(P_n; \varepsilon) \leq \text{polynomial in } n.$$
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Limited only by their statistical assumptions, these analyses demonstrated, at least conceptually, why the simplex algorithm typically behaves well and is efficient.
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The subsequent analysis by Spielman and Teng [2004] improved these analyses by providing estimates for randomly perturbed linear programs. This analysis has since been improved, see [Dadush and Huiberts [2020], Vershynin [2009], Deshpande and Spielman [2005]], for example.
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For any average-case analysis one hopes to continue to:

- Expand the class of distributions that can be considered.
- Increase the precision of the resulting estimates.
- Collect additional algorithms that can be analyzed with the same or similar techniques.
We also highlight two other success stories in average-case analysis. These are of a different flavor because randomization is introduced to algorithms to improve their performance. And subsequently, one has a natural distribution over which to compute averages, but the problem being solved is deterministic.
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The first algorithm is the power method with randomized starting. The power method is an algorithm to compute the dominant eigenvalue (provided it exists) of a matrix. It also approximates the dominant eigenvector.
Some history

The power method

1. $x_0$ is the initial vector, $\|x_0\| = 1$ and $W$ is given.
2. For $k = 1, 2, \ldots$
   
   2.1 Compute $v_k = Wx_{k-1}$
   
   2.2 Compute $\mu_k = v_k^T x_{k-1}$
   
   2.3 Compute $x_k = v_k / \|v_k\|

The iterates $\mu_k$, under mild assumptions, will converge to the dominant eigenvalue of $W$. It is well-known that the power method will converge at a exponential rate depending on the ratio of the largest-to-next-largest eigenvalue (a relative spectral gap).
If $W$ is positive semi-definite and $x_0$ is chosen randomly ($x_0 = \text{np.random.randn}(n)$, $x_0 \leftarrow x_0/\|x_0\|$), then it was shown in Kuczyński and Woźniakowski [1992] that a spectral gap is not need to get average-case error bounds of the form:

$$
\mathbb{E} \left[ \frac{|\mu_k - \lambda_{\text{max}}|}{|\lambda_{\text{max}} - \lambda_{\text{min}}|} \right]_{E_k(P_n;\text{Power})} \leq 0.871 \frac{\log n}{k - 1}.
$$

The power method can also be analyzed on random matrices, see Kostlan [1988], Deift and Trogdon [2017].
Lastly, a discussion that is closer to the heart of the matter is the work of Strohmer and Vershynin [2009] on the randomized version of the original Kaczmarz algorithm [Kaczmarz [1937]] for the solution of overdetermined linear systems.

The Kaczmarz Algorithm

1. $x_0$ is the initial vector and $A$ is given.
2. For $k = 1, 2, \ldots$
   2.1 Select a row $a_j$ of $A$ (add randomness here!)
   2.2 Compute $x_k = x_{k-1} - \frac{b_j - a_j^T x_{k-1}}{\|a_j\|^2} a_j$
Some history

For a consistent overdetermined system $Ax = b$ it was shown that the method satisfies

$$\mathbb{E} \underbrace{\|x_k - x\|^2}_{E_k(P_n;\text{Kaczmarz})} \leq \left(1 - \frac{1}{\kappa(A^TA)}\right)^k \|x_0 - x\|^2$$

where $\kappa(A^TA)$ is the condition number of $A^TA$ (to be discussed more later).
The power of random matrix theory (RMT) is that one can ask and answer more involved questions:

- If $P_n$ is drawn randomly, then

$$T_A(P_n, \varepsilon)$$

is an integer-valued random variable. While it is important bound its expectation or moments, what about its distribution as $n \to \infty$?

- With the same considerations

$$E_k(P_n; A)$$

is a random variable. Can we understand its distribution?
Sagun et al. [2017] present experiments to demonstrate that the halting time $T_{\text{SGD}}(P_n; \varepsilon)$ for a number of neural network architectures exhibits universality. That is, after proper centering and rescaling, the resulting statistics do not depend (in the limit) on the distribution on $P_n$. 
Universality

A wide variety of numerical algorithms have been demonstrated (both empirically and rigorously) to have universal halting times (i.e., runtimes, iteration counts, etc.). The study of universality in halting time was initiated by Pfrang et al. [2014] and broaded in Deift et al. [2014].

Universality in halting time is the statement that for a given $\mathcal{A}$, and a wide class of ensembles $\mathcal{E}$, there are constants $\mu = \mu(\mathcal{E}, \varepsilon, n)$ and $\sigma = \sigma(\mathcal{E}, \varepsilon, n)$ and $\varepsilon = \varepsilon(\mathcal{E}, n)$ such that

$$\lim_{n \to \infty} \mathbb{P}_\mathcal{E} \left( \frac{T_{\mathcal{A}}(P_n, \varepsilon) - \mu}{\sigma} \leq t \right) = F_{\mathcal{A}}(t).$$
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$$
\lim_{n \to \infty} \mathbb{P}_E \left( \frac{T_{\mathcal{A}}(P_n, \varepsilon) - \mu}{\sigma} \leq t \right) = F_{\mathcal{A}}(t).
$$

The limiting distribution is independent of the choice for $\mathcal{E}$. 
A case study: Regression
A natural first place to combine RMT and optimization/ML with a view toward universality is in the study of linear regression:

\[
\arg \min_{x \in \mathbb{R}^n} \left\{ \mathcal{L}(x) := \frac{1}{2d} \|Ax - b\|^2, \quad b = Aa + \eta \right\},
\]

where \(a\) is the signal, \(\eta\) is additive noise, and

\[A\] is a \(d \times n\) matrix.\]
A natural first step to combine RMT and optimization/ML with a view toward universality is in the study of linear regression:

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where \(a\) is the signal, \(\eta\) is additive noise, and

\[A\text{ is a } d \times n\text{ matrix.}\]

There are, of course, many iterative algorithms to solve this problem and we focus on two:

1. the conjugate gradient algorithm (CG) [Hestenes and Steifel [1952]], and
2. the gradient descent algorithm (GD).
The Conjugate Gradient Algorithm

1. $x_0$ is the initial guess.
2. Set $r_0 = A^T b - A^T Ax_0$, $p_0 = r_0$.
3. For $k = 1, 2, \ldots, n$
   3.1 Compute $a_{k-1} = \frac{r^*_{k-1}r_{k-1}}{r^*_{k-1}A^TAp_{k-1}}$.
   3.2 Set $x_k = x_{k-1} + a_{k-1}p_{k-1}$.
   3.3 Set $r_k = r_{k-1} - a_{k-1}A^TAp_{k-1}$.
   3.4 Compute $b_{k-1} = -\frac{r^*_k r_k}{r^*_{k-1}r_{k-1}}$.
   3.5 Set $p_k = r_k - b_{k-1}p_{k-1}$.
A natural first step

Why consider CG?

CG is a highly-structured algorithm with connections to the Lanczos iteration and the theory of orthogonal polynomials. While we do not discuss many of these details, they play an important role in the analysis. CG is also a method-of-choice in the broader computational mathematics community.

A simplification.

Instead of considering CG on the normal equations, \( A^T A x = A^T b \), we first consider a slightly simpler problem: CG applied to \( A^T A x = c \).
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CG applied to \( A^T Ax = c \).
Scaling regions

Scaling regions show up here in the relationship between $n$ and $d$ in a sample covariance matrix $W = \frac{A^TA}{d}$ ($A$ is $d \times n$).

**Scalings of sample covariance matrices**

- $d = \lfloor nr \rfloor$ for $r > 1$
- $d = n$
- $d = \lfloor n + cn^\alpha \rfloor$ for $0 < \alpha < 1$
Scaling regions show up here in the relationship between $n$ and $d$ in a sample covariance matrix $W = \frac{A^T A}{d}$ ($A$ is $d \times n$).

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$$\kappa(W) = \frac{\sigma_1(W)}{\sigma_n(W)},$$

i.e., the ratio of the largest to the smallest singular value of $W$.

Matrices with small condition numbers are said to be well conditioned while those with larger condition numbers are said to be ill conditioned.
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### Scalings of sample covariance matrices

- $d = [nr]$ for $r > 1$ (well conditioned)
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Three distinct behaviors of CG depending on scaling region

- CG with $d = 400$ and $n = 400$
- CG with $d = 800$ and $n = 400$
- CG with $d = 520$ and $n = 400$
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ill-conditioned
Three distinct behaviors of CG depending on scaling region

- **CG with d = 400 and n = 400**
  - *ill-conditioned*
  - Density vs. Halting time

- **CG with d = 800 and n = 400**
  - *well-conditioned*
  - Density vs. Halting time

- **CG with d = 520 and n = 400**
  - Density vs. Halting time
Three distinct behaviors of CG depending on scaling region

- **ill-conditioned**: When $d = 400$ and $n = 400$, the halting time distribution exhibits a sharp peak, indicating high ill-conditioning.

- **well-conditioned**: When $d = 800$ and $n = 400$, the distribution is more spread out, suggesting better conditioning.

- **"in between"**: When $d = 520$ and $n = 400$, the distribution lies somewhere in between the extreme cases, indicating intermediate behavior.

These visualizations demonstrate how changes in scaling affect the convergence properties of the CG algorithm.
Qualitative comparison with SGD

While the mechanisms behind these behaviors are surely different, we see a non-trivial histogram in each setting.

For CG on Wishart matrices, it can be shown that

$$\|r_k\| = \|c - A^T A x_k\|^\text{dist} = \prod_{j=0}^{k-1} \frac{\chi_{n-j-1}}{\chi_{d-j}},$$

for independent chi-distributed random variables.
So, if we set
\[ E_r(\text{Wishart}; \text{CG}) = \|r_r\| \]
we can analyze the halting time to see that
\[
T_{\text{CG}}(\text{Wishart}, \varepsilon) \approx \frac{2}{C} n^{1-\alpha} \log \varepsilon^{-1} + O(n^{3/2-2\alpha})N(0, 1),
\]
for \(1/2 < \alpha < 1\).
It turns out that the errors $E_r(P_N; A)$ for iterative methods for a linear system involving $A^T A$ are often analyzable in the well-conditioned, ill-conditioned and “in between” regimes. But the analysis of the halting time can be much more involved because the halting time $T_A(P_n, \varepsilon)$ can tend to infinity with $n$!
So, we, for the time being, let $d = \lfloor nr \rfloor$ for $r > 1$. 
The Gradient Descent Algorithm

1. $x_0$ is the initial vector.
2. For $k = 1, 2, \ldots$
   2.1 Select step size $\gamma_k$
   2.2 Compute $x_k = x_{k-1} - \gamma_k \nabla L(x_{k-1})$

Recall that the gradient of the regression functional is
$\nabla L(x) = Wx - c$, $W = A^T A$. A direction calculation reveals that $x_k = Q_k(Wc)$, for a polynomial $Q_k$ of degree $k-1$ with coefficients that depend on $\gamma_j$, $j = 1, 2, \ldots, k$. 
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A direction calculation reveals that

$$x_k = Q_k(W)c,$$

for a polynomial $Q_k$ of degree $k - 1$ with coefficients that depend on $\gamma_j, j = 1, 2, \ldots, k.$
For simplicity, suppose that $\mathbf{W}$ is full rank. Then if $\mathbf{x}$ is the true minimizer, a crucial calculation is that

$$\mathbf{x} - \mathbf{x}_r = \mathbf{W}^{-1} \mathbf{c} - \mathbf{Q}_r(\mathbf{W})\mathbf{c} = \mathbf{W}^{-1} (\mathbf{I}_n - \mathbf{WQ}_r(\mathbf{W})) \mathbf{c}. \quad \underset{R_k(\mathbf{W})}{\mathbf{R}_k(\mathbf{W})}$$

Note that $R_k$ is a polynomial of degree $k$ satisfying $R_k(0) = 1.$

Then

$$\nabla \mathcal{L}(\mathbf{x}_r) = \mathbf{Wx}_r - \mathbf{Wx} = R_k(\mathbf{W})\mathbf{c},$$

$$\|R_k(\mathbf{W})\mathbf{c}\|^2 = \mathbf{c}^T R_k(\mathbf{W})^2 \mathbf{c}.$$
Example: GD

For GD follows that the difference $x_k - x$ satisfies

$$x_k - x = x_{k-1} - x - \gamma_k (Wx_{k-1} - Wx) = (I_n - \gamma_k W)(x_{k-1} - x).$$

And so,

$$R_k(x) = \prod_{j=1}^{k} (1 - \gamma_j x).$$
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And so,

$$R_k(x) = \prod_{j=1}^{k} (1 - \gamma_j x).$$

For CG the polynomial $R_k$ is best characterized using the theory of orthogonal polynomials.
\[ \|R_k(W)c\|^2 = c^T R_k(W)^2 c \]

The error analysis of GD (and, as it turns out, CG) is reduced to:

1. The determination/characterization of the polynomial \( R_k \).
2. The estimation of \( c^T R_k(W)^2 c \).
For many methods of interest (CG and GD included), the coefficients of $R_k$ depend continuously on the eigenvalues and eigenvectors of $W$ in a sufficiently strong sense that

$$R_k(x) \xrightarrow{n \to \infty} \mathcal{R}_k(x) \leftarrow \text{deterministic}.$$ 

Then, one can conclude

$$c^T R_k(W)^2 c \xrightarrow{n \to \infty} \int_{\mathbb{R}} \mathcal{R}_k(x)^2 \mu_{\text{SCM}}(dx).$$

This provides a deterministic limit for the (random) errors that are encountered throughout the algorithm.

Note: This is true only if $c$ is independent of $W$ and in the regression problem it is not.
Building back to true regression

For the regression problem, we have

\[ c = \frac{1}{n} \left[ A^T a + A^T \eta \right]. \]

Then

\[ \| \mathcal{L}(x_k) \|^2 = a^T W^2 R_k(W)^2 a + \frac{1}{n^2} \eta^T A R_k(W)^2 A^T \eta + \frac{2}{n} a^T W R_k(W)^2 A^T \eta \]
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$$\begin{align*}
\lim_{n \to \infty} \Pr & \left[ R \int_{\mathbb{R}} x^2 R_k(x)^2 \mu_{SCM}(dx) + \tilde{R} \int_{\mathbb{R}} x R_k(x)^2 \mu_{SCM}(dx) \right] \\
& = \epsilon_k^2
\end{align*}$$

Important features:

• This demonstrates that the entire spectrum of $W$ contributes via $\mu_{SCM}$

• Nearly all probabilistic analyses of algorithms give inequalities whereas this gives true leading-order behavior.
Building back to true regression

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\[ \approx 0 \quad \text{if} \quad a, \eta \text{ indep.} \]

\[ \Pr_{n \to \infty} \int_\mathbb{R} x^2 \mathcal{R}_k(x)^2 \mu_{SCM}(dx) + \tilde{R} \int_\mathbb{R} x \mathcal{R}_k(x)^2 \mu_{SCM}(dx). \]

Important features:

- This demonstrates that the entire spectrum of \( W \) contributes via \( \mu_{SCM} \)
- Nearly all probabilistic analyses of algorithms give inequalities whereas this gives true leading-order behavior.
Step-size selection

\[ \| \mathcal{L}(x_k) \|^2 \xrightarrow{n \to \infty} \mathbb{P}_n \mathcal{R} \int_{\mathbb{R}} x^2 \mathcal{R}_h(x)^2 \mu_{SCM}(dx) + \tilde{\mathcal{R}} \int_{\mathbb{R}} x \mathcal{R}_h(x)^2 \mu_{SCM}(dx) \]

If one has a good guess as to what the limiting distribution \( \mu_{SCM} \) is then the \( \gamma_k \)'s in GD can be chosen based on this limit — to minimize this expression, see Pedregosa and Scieur [2020].
Step-size selection

\[ \| \mathcal{L}(x_k) \|^2 \xrightarrow{\text{Pr}} R \int_{\mathbb{R}} x^2 \mathcal{R}_k(x)^2 \mu_{SCM}(dx) + \tilde{R} \int_{\mathbb{R}} x \mathcal{R}_k(x)^2 \mu_{SCM}(dx) \]

If one has a good guess as to what the limiting distribution \( \mu_{SCM} \) is then the \( \gamma_k \)'s in GD can be chosen based on this limit — to minimize this expression, see Pedregosa and Scieur [2020]. Furthermore, by preconditioning one can make such a guess valid, see Lacotte and Pilanci [2020].
Provided that $e_k \xrightarrow{k \to \infty} 0$, one finds that

$$\lim_{n \to \infty} P(T_A(P_n; \varepsilon) = \min\{k : e_k < \varepsilon\}) = 1,$$

for most choices of $\varepsilon$.

This turns out to be true for all $d \geq n$, $n \to \infty$, for the regression problem with CG or GD.
Deterministic halting for CG with $r = 2, \varepsilon = 10^{-4}$
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Deterministic halting for CG with $r = 2, \varepsilon = 10^{-4}$
RMT provides non-trivial tractable models to analyze the statistics of optimization algorithms.

Other algorithms are analyzable:

- MINRES algorithm
- Polyak algorithm
- Nesterov accelerated algorithm
- SGD for regression
- ...

See the preprints: Paquette and Trogdon [2020], Paquette et al. [2021], Ding and Trogdon [2021], Paquette et al. [2020]
Other ensembles are analyzable using the following results from RMT:

- Spiked random matrices (see Baik et al. [2005], Bloemendal and Virág [2013], Ding and Yang [2019], and many more)
- Nonlinear models (see Part 4)
- Random graphs (see Erdős et al. [2013], for example)
- Invariant ensembles (see Bourgade et al. [2014], Deift [2000] and many more)
Open questions

Many open questions remain:

• To what extent can one move these ideas beyond regression? To a two-layer network? Rank-one matrix completion problem?

• What is a good probability distribution to study? Wishart is clearly the place to start but what is relevant in practice?
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See Colab for a CG demo

https://colab.research.google.com/drive/1UZRSK665b8qq0NQFwMCwrVabP1B-7nK?usp=sharing
References


